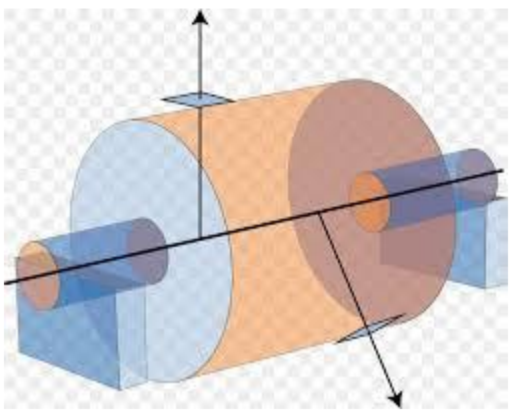
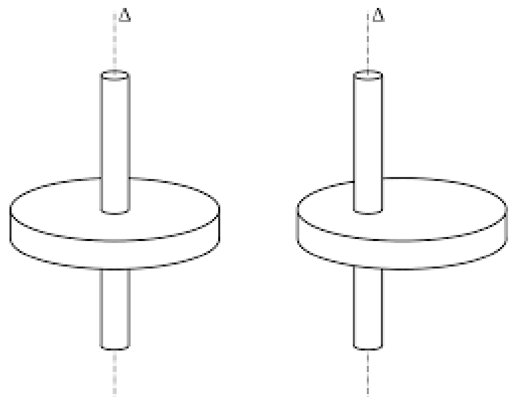
COLLEGE OF ENGINEERING & TECHNOLOGY

Critical Speeds of Shafts





Subject:- DOM
Code:-3151911

Prepared by:
Asst.Prof.Chirag Mevada
(Mechanical Department,ACET)

COLLEGE OF ENGINEERING & TECHNOLOGY

Critical Speed of Shaft Carrying Single Rotor (Without Damping)

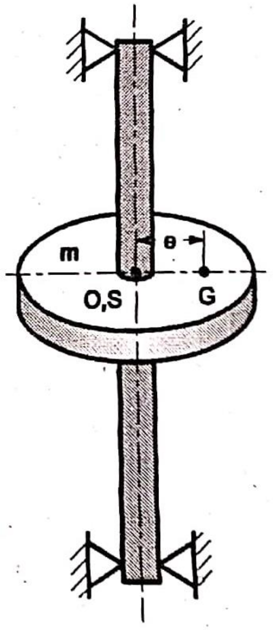
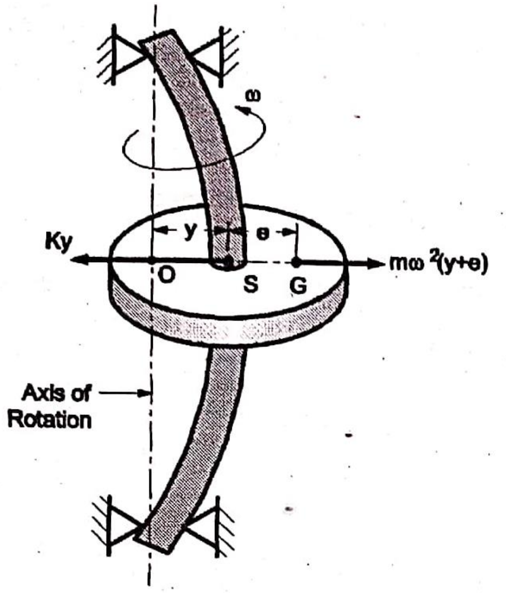


Fig. 13.2.1 (a) Shaft in Stationary Condition



(b) Shaft in Rotating Condition
Fig. 13.2.1

– When shaft is in rotating condition as shown in Fig. 13.2.1(b), then there are two forces acting on the shaft :

1. **Centrifugal Force = m ω^2 (y + e) :** It acts in radially outward direction through point G.
2. **Restoring Force = Ky :** It acts in radially inward direction through point G.

– In equilibrium condition, the centrifugal force is equal to restoring force. Therefore,

$$\text{Centrifugal force} = \text{Restoring force}$$

$$\therefore \; m\omega^2 (y + e) = Ky$$

$$\therefore \; m\omega^2 y + m\omega^2 e = Ky$$

$$Ky - m\omega^2 y = m\omega^2 e$$

$$y(K - m\omega^2) = m\omega^2 e$$

$$\therefore \; y = \frac{m\omega^2 e}{K - m\omega^2} = \frac{\frac{m\omega^2 e}{K}}{1 - \frac{m\omega^2}{K}} = \frac{\frac{\omega^2 e}{K/m}}{1 - \left(\frac{\omega^2}{K/m}\right)}$$

$$\text{or} \quad y = \frac{\left(\frac{\omega}{\omega_n}\right)^2 e}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \qquad \left[\because \omega_n^2 = \frac{K}{m}\right]$$

...(13.2.1)

- From Equation (13.2.1) it is clear that, as the angular speed of the shaft 'ω' increases, the deflection of the shaft 'y' increases. When 'ω' becomes equal to 'ω_n', the deflection of the shaft y becomes infinity.
- Thus, the speed at which the deflection of the shaft tends to be infinity is known as **critical speed** or **whirling speed.**
- Therefore, the critical speed or whirling speed of shaft is given by,

$$\left.\begin{aligned} \omega_c &= \omega_n \\ \text{or} \quad \omega_c &= \sqrt{\frac{K}{m}}, \text{ rad/s} \\ \text{or} \quad \omega_c &= \sqrt{\frac{g}{\delta}}, \text{ rad/s} \end{aligned}\right\} \quad \text{...(13.2.2)}$$

$$\text{or} \quad N_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

$$\text{or} \quad N_c = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} \text{ r.p.s.} \quad \text{...(13.2.3)}$$

Where, N_c = Critical speed, in r.p.s.

δ = Static deflection of the shaft, m

Hence, Equation (13.2.1) can be written as,

$$y = \frac{(\omega/\omega_c)^2 e}{1-(\omega/\omega_c)^2} \quad \text{...(13.2.4)}$$

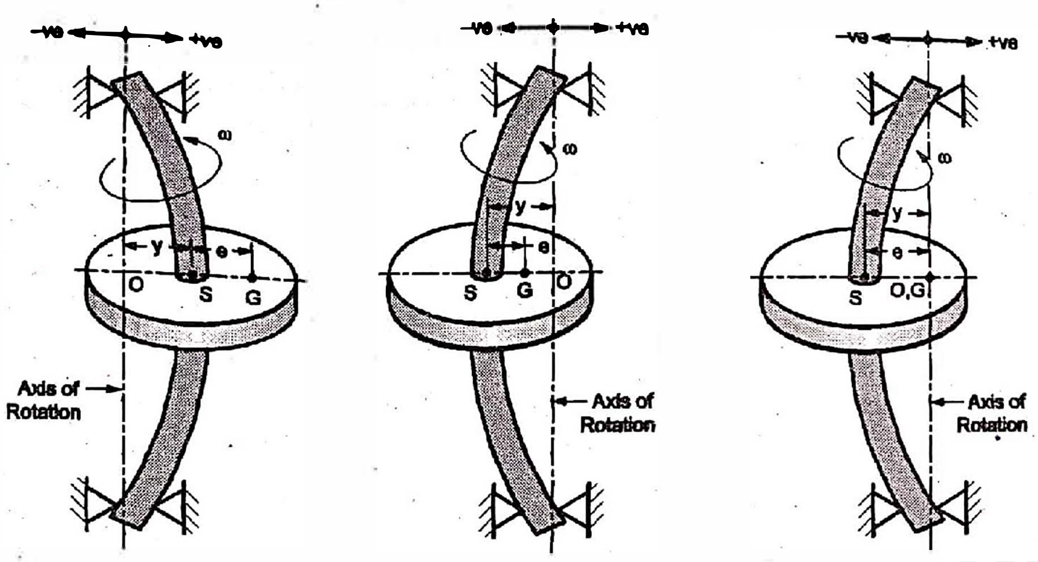
13.2.1 Ranges of Shaft Speed

- From Equation (13.2.4) it is seen that, there are three ranges of shaft speed 'ω' :

Ranges of shaft speed

1. Shaft speed (ω) < Critical speed (ω_c)
2. Shaft speed (ω) = Critical speed (ω_c)
3. Shaft speed (ω) > Critical speed (ω_c)

Fig. C13.1 : Ranges of shaft speed



(a) When ω < ω_c **(b) When ω > ω_c** **(c) When ω >> ω_c**

Critical Speed of Shaft Carrying Single Rotor (With Damping)

1. **Centrifugal Force = $m\omega^2 a$:** It is acting at point 'G' along OG.
2. **Restoring Force = Ky :** It is acting at point 'S' along SO.
3. **Damping Force = $c\omega y$:** It is acting at point 'S' in a direction opposite to the linear velocity of point 'S'.

   where, ωy = the linear velocity of point 'S', m/s

- Due to damping force, the points O, S and G are no longer on a straight line, but they form a triangle as shown in Fig. 13.3.1(b).
- For the forces acting on the shaft, at equilibrium,

$$\Sigma F_x = 0;$$

$$\therefore \quad -Ky + m\omega^2 a \cos\psi = 0 \qquad \text{...(a)}$$

and

$$\Sigma F_y = 0;$$

$$\therefore \quad -c\omega y + m\omega^2 a \sin\psi = 0 \qquad \text{...(b)}$$

- It is essential to eliminate 'ψ' and 'a' from Equations (a) and (b).
- From Fig. 13.3.2 ;

$$a \sin\psi = e \sin\phi \qquad \text{...(c)}$$

and

$$a \cos\psi = y + e\cos\phi \qquad \text{...(d)}$$

- On substituting the values of '**a cos ψ**' from Equation (d) in Equation (a) we get,

$$-Ky + m\omega^2 (y + e\cos\phi) = 0$$

$$-Ky + m\omega^2 y + m\omega^2 e\cos\phi = 0$$

$$(K - m\omega^2)\, y = m\omega^2 e \cos\phi \qquad \text{...(e)}$$

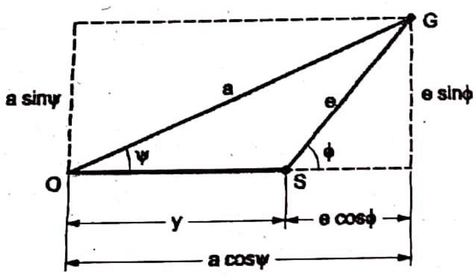


Fig. 13.3.2

- On substituting the value of a sin ψ from Equation (c) in Equation (b), we get,

$$-c\omega y + m\omega^2 e \sin\phi = 0$$

$$\therefore \quad c\,\omega\, y = m\omega^2 e \sin\phi \qquad \text{...(f)}$$

- Dividing Equation (f) by Equation (e), we get,

$$\tan\phi = \frac{c\omega}{(K - m\omega^2)} \qquad \text{...(13.3.1)}$$

or

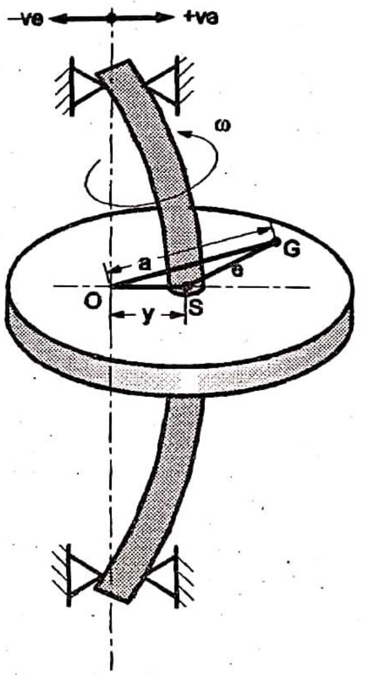
$$\tan\phi = \frac{2\xi(\omega/\omega_n)}{1 - (\omega/\omega_n)^2} \qquad \text{...(13.3.2)}$$

$$\therefore \quad \phi = \tan^{-1}\left[\frac{2\xi(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}\right]$$

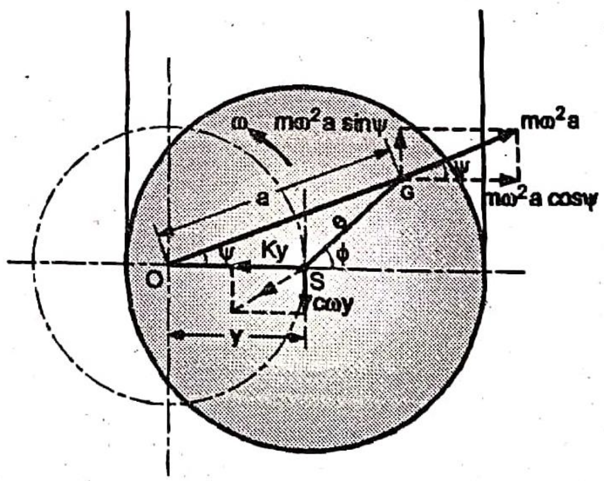
or

$$\phi = \tan^{-1}\left[\frac{2\xi(\omega/\omega_c)}{1 - (\omega/\omega_c)^2}\right] \qquad \text{...(13.3.3)}$$

- Equation (13.3.2) gives the phase angle between the eccentricity line and deflection line i.e. angle by which deflection lags. Equation (13.3.1) may be represented in the vector form as shown in Fig. 13.3.3.



(a) Deflected Position of Shaft



(b) Forces Acting on Shaft in Deflected Position

Fig. 13.3.1

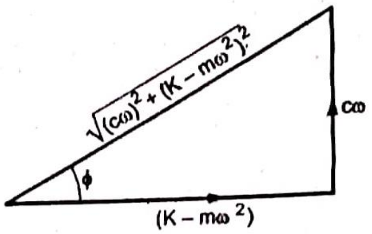


Fig. 13.3.3

– From Fig. 13.3.3 we can also write,

$$\sin\phi = \frac{c\omega}{\sqrt{(c\omega)^2 + (K - m\omega^2)^2}} \qquad \text{...(g)}$$

– Substituting the value of $\sin\phi$ from Equation (g) in Equatio (f), we get,

$$c\,\omega y = m\omega^2 e \frac{c\omega}{\sqrt{(c\omega)^2 + (K - m\omega^2)^2}}$$

$$\therefore \quad y = \frac{m\,\omega^2 e}{\sqrt{(K - m\omega^2)^2 + (c\omega)^2}}$$

$$\therefore \quad y = \frac{e\,(\omega/\omega_n)^2}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \qquad \text{...(13.3.4}$$

or

$$y = \frac{e\,(\omega/\omega_c)^2}{\sqrt{[1 - (\omega/\omega_c)^2]^2 + \left(2\xi\frac{\omega}{\omega_c}\right)^2}}$$

– Equation (13.3.4) gives the deflection of the geometric center of the rotor from the initial undeflected position.

– It is seen that Equations (13.3.3) and (13.3.4) for whirling of shaft are same as that for the forced damped vibrations with rotating and reciprocating unbalance. In case of forced damped vibrations due to rotating or reciprocating unbalance, the unbalance was in terms of the small mass m_o, whereas in this case the unbalance is defined in terms of the total mass 'm' with eccentricity 'e'.

13.3.1 Various Possible Phase Angles

– From Equation (13.3.3) following observations are made :

(i) When $\omega << \omega_n$; $\phi \simeq 0°$ (Heavy side out)

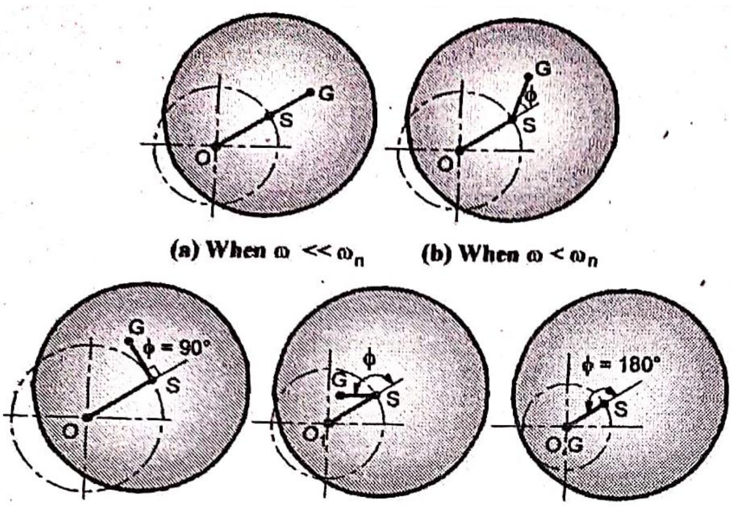
(ii) When $\omega < \omega_n$; $0° < \phi < 90°$ (Heavy side out)

(iii) When $\omega = \omega_n$; $\phi = 90°$

(iv) When $\omega > \omega_n$; $90° < \phi < 180°$ (Light side out)

(v) When $\omega >> \omega_n$; $\phi \simeq 180°$ and $y \simeq -e$ (Light side out and rotor rotates about its C.G.)

– Fig. 13.3.4 shows various possible phase angles with damping.



(a) When $\omega << \omega_n$ (b) When $\omega < \omega_n$

(c) When $\omega = \omega_n$ (d) When $\omega > \omega_n$ (e) When $\omega >> \omega_n$ and $y = -e$

Fig. 13.3.4 : Possible Phase Angles (With Damping)

Secondary Critical Speed

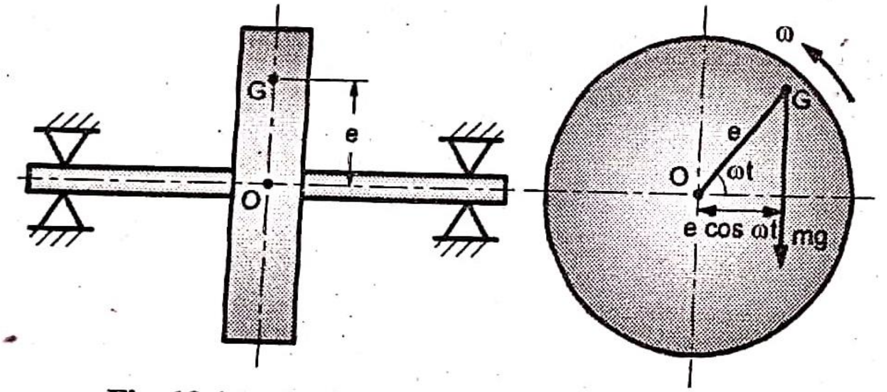
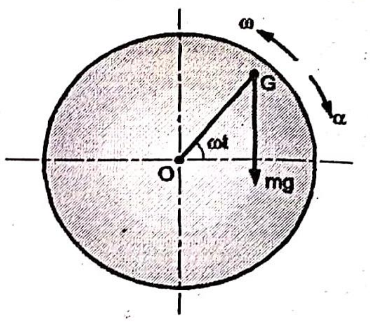
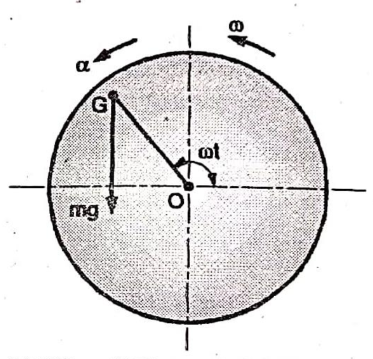


Fig. 13.4.1 : Horizontal Shaft With Single Rotor



(a) When G Lies on Right Side of O



(b) When G Lies on Left Side of O
Fig. 13.4.2

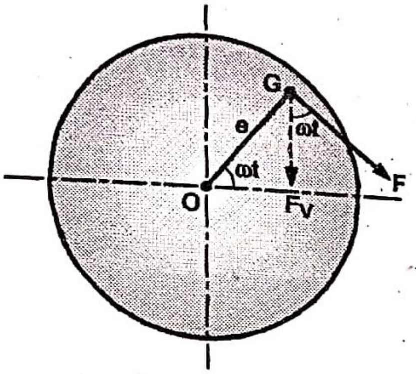


Fig. 13.4.3

- The magnitude of the varying torque acting on the rotor due to gravitational force is given by,

$$T = m\,g\,e\cos\omega t \qquad ...(a)$$

- Due to this varying torque T, the rotor will retard (from $\omega t = -90°$ to $+90°$) and accelerate (from $\omega t = +90°$ to $270°$) during rotation.
- The angular acceleration or retardation of the rotor is given by,

$$\alpha = \frac{T}{I} = \frac{m\,g\,e\cos\omega t}{I} \qquad ...(b)$$

where, I = mass moment of inertia of rotor, kg-m^2

- Hence, the tangential acceleration or retardation of the point G is given by,

$$f_t = \alpha \cdot e = \frac{m\,g\,e^2\cos\omega\,t}{I} \qquad ...(c)$$

- Hence the tangential force acting at point G is given by,

$$F = m \cdot f_t = \frac{m^2\,g\,e^2\cos\omega\,t}{I} \qquad ...(d)$$

- From Fig. 13.4.3,
- The vertical component of the tangential force is given by,

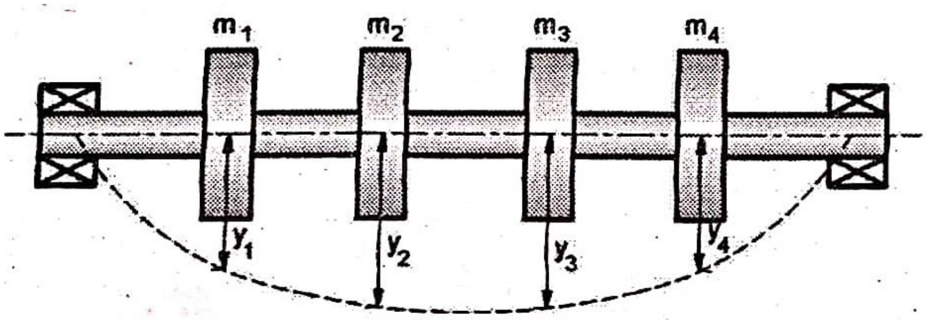
$$F_v = F\cos\omega t = \frac{m^2\,g\,e^2\cos^2\omega\,t}{I}$$

$$= \frac{m^2\,g\,e^2}{2I} \times 2\cos^2\omega\,t$$

$$= \frac{m^2\,g\,e^2}{2I}\,(1 + \cos 2\omega\,t)$$

...[$\because 2\cos^2\theta = (1 + \cos 2\theta)$]

Critical Speed of Shaft Carrying Multiple Rotors



1. Rayleigh's Method

- This method is based on the principle that, the maximum kinetic energy is equal to the maximum potential energy.
- The maximum kinetic energy is given by,

$$(KE)_{max} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_3 v_3^2 + \frac{1}{2} m_4 v_4^2 \ldots$$

$$\therefore (KE)_{max} = \frac{1}{2} m_1 (y_1 \omega_c)^2 + \frac{1}{2} m_2 (y_2 \omega_c)^2 + \frac{1}{2} m_3 (y_3 \omega_c)^2 + \frac{1}{2} m_4 (y_4 \omega_c)^2 + \ldots$$

$$\text{or} \quad (KE)_{max} = \frac{1}{2} \omega_c^2 \sum_{i=1}^{n} m_i y_i^2 \qquad \text{...(a)}$$

- The maximum potential energy is given by,

$$(PE)_{max} = \frac{1}{2} m_1 g y_1 + \frac{1}{2} m_2 g y_2 + \frac{1}{2} m_3 g y_3 + \frac{1}{2} m_4 g y_4 + \ldots$$

$$\text{or} \quad (PE)_{max} = \frac{1}{2} g \sum_{i=1}^{n} m_i y_i \qquad \text{...(b)}$$

- According to Rayleigh's method

$$(KE)_{max} = (PE)_{max}$$

$$\therefore \frac{1}{2} \omega_c^2 \sum_{i=1}^{n} m_i y_i^2 = \frac{1}{2} g \sum_{i=1}^{n} m_i y_i$$

$$\therefore \omega_c = \sqrt{\frac{g \sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i y_i^2}}, \text{ rad/s} \qquad \text{...(c)}$$

- Therefore, the natural frequency is,

$$\therefore f_c = \frac{\omega_c}{2\pi}$$

$$= \frac{1}{2\pi} \sqrt{\frac{g \sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i y_i^2}}, \text{ Hz} \qquad \text{...(13.5.1)}$$

AMIRAJ COLLEGE OF ENGINEERING & TECHNOLOGY

Critical Speed of Shaft Carrying Multiple Rotors

2. Dunkerley's Method

$$\frac{1}{(\omega_c)^2} = \frac{1}{(\omega_{c1})^2} + \frac{1}{(\omega_{c2})^2} + \frac{1}{(\omega_{c3})^2} + \frac{1}{(\omega_{c4})^2} + \dots + \frac{1}{(\omega_{cs})^2} \quad \dots(13.5.2)$$

where, ω_c = Critical speed of shaft carrying number of point loads and uniformly distributed load, rad/s

ω_{c1} = Critical speed of shaft neglecting its mass and all points loads except point load 1, rad/s

ω_{c2} = Critical speed of shaft neglecting its mass and all point loads except point load 2, rad/s

ω_{c3} = Critical speed of shaft neglecting its mass and all point loads except point load 3, rad/s

ω_{c4} = Critical speed of shaft neglecting its mass and all point loads except point load 4, rad/s

ω_{cs} = Critical speed of shaft considering its mass and neglecting all point loads, rad/s

– Consider a shaft carrying several point loads with uniformly distributed load, as shown in Fig. 13.5.2.

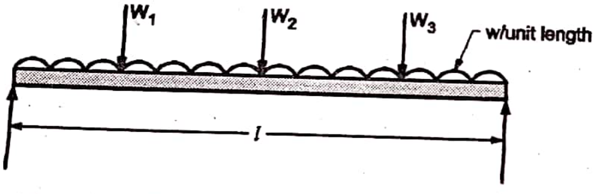


Fig. 13.5.2 : Shaft Carrying Several Point Loads With Uniformly Distributed Load

– Let,

$\delta_1, \delta_2, \delta_3$ = static deflections of shaft due to load W_1, W_2 and W_3 when considered separately

δ_s = static deflection of shaft due to self weight or due to the uniformly distributed load.

– We know that, natural frequency of transverse vibrations or critical speed due to load W_1 is,

$$\omega_{c1} = \sqrt{\frac{g}{\delta_1}}, \text{ rad/s}$$

– Similarly, natural frequency of transverse vibrations or critical speed due to loads W_2 and W_3 are,

$$\omega_{c2} = \sqrt{\frac{g}{\delta_2}}, \text{ rad/s}$$

and $$\omega_{c3} = \sqrt{\frac{g}{\delta_3}}, \text{ rad/s}$$

– Also the natural frequency of transverse vibrations or critical speed due to uniformly distributed load or self weight of the shaft is,

$$\omega_c = \sqrt{\frac{g}{\delta_s}}, \text{ rad/s}$$

AMIRAJ COLLEGE OF ENGINEERING & TECHNOLOGY

Notes

- The static deflection due to point load for a simply supported beam is given by,

$$\delta = \frac{W l_1^2 l_2^2}{3 EI l}$$

- The static deflection due to uniformly distributed load for a simply supported beam is given by,

$$\delta = \frac{5W l^4}{384 EI}$$

where, l_1 and l_2 = distance of point load from both ends

E = modulus of elasticity for the shaft material, N/m^2

I = moment of inertia of shaft $= \frac{\pi}{64} d^4$, m^4

l = total length of shaft, m

Ex. 13.5.2 GTU - May 14

A shaft of 50 mm diameter and 3 m length has a mass of 10 kg per meter length. It is simply supported at the ends and carries three masses of 70 kg, 90 kg and 50 kg at 1 m, 2 m and 2.5 m respectively from the left support. Find the natural frequency of transverse vibrations. Assume $E = 200 \times 10^9 N/m^2$.

Soln. : Given : Diameter of shaft,

$$d = 50 \text{ mm} = 0.05 \text{ m}$$

length of shaft, $l = 3$ m

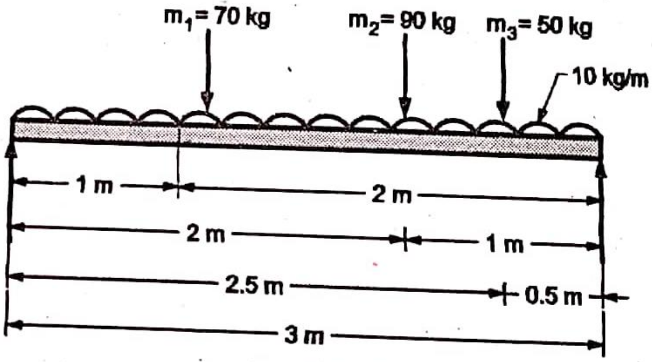


Fig. P. 13.5.2

1. **Static Deflections :** Refer Fig. P. 13.5.2;

- The static deflection due to mass m_1 is,

$$\delta_1 = \frac{m_1 g l_1^2 l_2^2}{3 EI l} = \frac{70 \times 9.81 \times (1)^2 \times (2)^2}{3 \times 2 \times 10^{11} \times \frac{\pi}{64}(0.05)^4 \times 3}$$

$$= 4.97 \times 10^{-3} \text{ m}$$

- The static deflection due to mass m_2 is,

$$\delta_2 = \frac{m_2 g l_1^2 l_2^2}{3 EI l}$$

$$= \frac{90 \times 9.81 \times (2)^2 \times (1)^2}{3 \times 2 \times 10^{11} \times \frac{\pi}{64}(0.05)^4 \times 3}$$

$$= 6.39 \times 10^{-3} \text{ m}$$

- The static deflection due to mass m_3 is,

$$\delta_3 = \frac{m_3 g l_1^2 l_2^2}{3 EI l}$$

$$= \frac{50 \times 9.81 \times (2.5)^2 \times (0.5)^2}{3 \times 2 \times 10^{11} \times \frac{\pi}{64}(0.05)^4 \times 3}$$

$$= 1.38 \times 10^{-3} \text{ m}$$

– The static deflection due to self weight is,

$$\delta_s = \frac{5 \text{ mg } l^4}{384 \text{ EI}} = \frac{5 \times 10 \times 9.81 \times (3)^4}{384 \times 2 \times 10^{11} \times \frac{\pi}{64}(0.05)^4}$$

$$= \mathbf{1.68 \times 10^{-3} \text{ m}}$$

2. Natural Frequencies Due to Individual Loads

– The natural frequency of transverse vibrations due to individual masses are,

$$\omega_{c1} = \sqrt{\frac{g}{\delta_1}} = \sqrt{\frac{9.81}{4.97 \times 10^{-3}}} = 44.43 \text{ rad/s}$$

$$\omega_{c2} = \sqrt{\frac{g}{\delta_2}} = \sqrt{\frac{9.81}{6.39 \times 10^{-3}}} = 39.18 \text{ rad/s}$$

$$\omega_{c3} = \sqrt{\frac{g}{\delta_3}} = \sqrt{\frac{9.81}{1.38 \times 10^{-3}}} = 84.31 \text{ rad/s}$$

$$\omega_{cs} = 1.1257\sqrt{\frac{g}{\delta}} = 1.1257\sqrt{\frac{9.81}{1.68 \times 10^{-3}}}$$

$$= 86.01, \text{ rad/s}$$

According to Dunkerley's method, .

$$\frac{1}{(\omega_c)^2} = \frac{1}{(\omega_{c1})^2} + \frac{1}{(\omega_{c2})^2} + \frac{1}{(\omega_{c3})^2} + \frac{1}{(\omega_{c4})^2} + \dots + \frac{1}{(\omega_{cs})^2}$$

$$= \frac{1}{(44.93)^2} + \frac{1}{(39.18)^2} + \frac{1}{(84.31)^2} + \frac{1}{(86.01)^2}$$

$$\therefore \frac{1}{(\omega_c)^2} = 1.43 \times 10^{-3}$$

$$\therefore \omega_c = 26.4 \text{ rad/s}$$

or $\mathbf{f_c = 4.202 \text{ Hz}}$ **...Ans.**